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Summary Sheet

A Better Solution for Merging Zone

Summary

It is common to experience slow-down when drivers go trough tollbooths and try to "fan out" from toll plazas. And the problem worsen in some states with dense turnpike network. So a new design of merging zone is desperately needed.

In this paper, we focus on figuring out the problem by finding a solution including size, shape, and merging pattern, our goal is to make our solution more outstanding than those in common use. Some basic assumptions is made to simplified our modelling process. And we develop a series of sub-models before the better solution comes into being – a Markov process model to simulate car flow and lane situation, an OD estimate model to better predict Markov procession's state space , a model to depict autonomous car and some other models including accident prevention, throughput, and cost. We come up with assessment to determine which solution is the best based on the shape, size and merging pattern.

We contrive three merging patterns, divide merging zone into three kinds of shape and study two types of size and test their influence on our models respectively.

Our better solution is a combination of best results about size, shape, and merging pattern. And we add new features in it by using dislocated merging zone units, which means tollbooths in our solution are no longer aligned, and this is why we name it $Dislocated\ Merging\ Zone$. We measure our solution in light and heavy traffic, and it performs better than common-used solution and its superiority remains firmed especially in heavy traffic.

And we make extension in our solution. We figure out how our solution changed by autonomous vehicles addition and different kinds of tollbooths. These results prove the rationality of our better solution.

Admittedly, our model ignores variety of cars, drivers' physiologic condition and outer factors like weather and topography. One pleasant thing we find is insensitivity of our model towards traffic change, especially in heavy traffic.

Based on all those test results from the models above, we can conclude with confidence that our model can help ease slow-down problem of toll plaza in the future, and it owns a advantage of low demand for vast land. And we believe it is practical in some states with dense turnpike network like New Jersey.

Keywords: Markov process model, cars model, merging zone

Contents

1 Introduction

1.1 Background

Many highways around the globe require tolls at certain intervals to pay for necessary infrastructure and road repair. But when the arriving traffic flow exceeds the "fan out" capacity, congestion forms, and thus reduce the outflow due to the congestion-induced capacity drop. [1]

Traffic congestion levels have been increasing significantly in the last few decades, and this trend seems to continue. The general perception is that crash frequency increases with increasing congestion levels.[2]

Chicago tollway officials are studying whether it makes sense to remove some toll plazas as a way to ease congestion.[3] Another way to solve the problem is redesigning new merging zone to improve "fan out" capacity.

Many studies have been done on merging zone design, when finding a solution of merging zone, size, shape and merging pattern should be taken into account. With more self-driving car adding to traffic mix, the merging zone should be more flexible to adapt to the features of self-driving cars.

1.2 Problem Restatement

We are required to find a better solution about the shape, size, and merging pattern of the area following the toll barrier.

Our tasks include:

- Determine the performance of our solution in light and heavy traffic.
- How to change our solutions more autonomous vehicles are added to the traffic mix.
- Flexibility of our solution affected by the proportions of different kinds of tollbooths.

The problem is analysed into several parts:

- Create a mathematical model to describe the toll in the question.
- Test several isolated models to integrate our better solution.
- Adjust parameters to adapt for different situations.

2 Assumption

2.1 Cars/Drivers

- **Cars are indistinguishable.** All cars have the same length and the same maximum speed.

- **Cars are generated according to a probability distribution.** We start them at 1 meter from the tollbooth and generate for a fixed amount of simulated time (usually about 1 h), then keep running until all have gotten to the end of the simulated road 1 mi beyond the tollbooth. There are no entry or exit ramps in the 1 mi section leading to the tollbooth. Some vehicles are classified as trucks, which function identically but must use manual tollbooths if they do not have an electronic pass.

- **Cars pack closely in a tollbooth line.** Drivers don't want people from other lanes to cut into their line, so they follow at distances closer than suggested on state driver's license exams.

- **Drivers' physiologic condition is negligible.** Because we focus on the merging zone itself, drivers' physiologic condition like reaction time and health condition will not be considered in accident prevention.

2.2 Lanes

- **The traffic situation is not affected by external factors.** Such factors include environment, climate, and so on.

- **Two-way highways are equivalent to two independent highways.** We consider only divided highways.

- **The toll plazas are not near on-ramps or exits.** We do not consider the possibility of additional cars merging, only those that were already on the main road.

- **Lanes are smooth.** There are no traffic roadblocks on the road.

3 Terminologies and Notations

3.1 Notations Table

Table 1: Notations

Symbol	Description
A	Accident factor
B	Tollbooths number
C	Cost
D	Delay time
T	Throughput
L	Lanes number
Q	Queue length
S	Size
θ	Angle formed by tollbooth lane and highway lane in merging zone

3.2 Terminologies

Traffic Condition We only consider two kinds of traffic condition in this problem. Light traffic is when T is under 3000 per hour. Accordingly, heavy traffic means T is greater than 3000 per hour.

Tollbooth The lane in the toll plaza between the entry highway and the exit highway, each contain a tollbooth.[4]

Merging Zone The B-lane region of the toll plaza between the tollbooth and the exit high way.

Merging Zone Unit A merging zone whose number of highway lane is set as 1. It is convenient for model test.

Size Ratio of highway lane number to tollbooth lane number, that is,

$$S = \frac{L}{B} \tag{1}$$

Shape We define shape as the angle θ formed by tollbooth lane and highway lane in merging zone. And we suggest to rate shape into the following level(higher level number represents shaper shape):

Table 2: Different Level of Shape

Level	1	2	3
θ	$[0, \frac{\pi}{20}]$	$(\frac{\pi}{20}, \frac{\pi}{10}]$	$(\frac{\pi}{10}, \frac{\pi}{2}]$
Description	Smooth	Normal	Sharp

According to design methods of highway, $\tan\theta$ is recommended to be designed under $\frac{1}{3}$.

Merging Pattern The way several lanes merge into one[4].

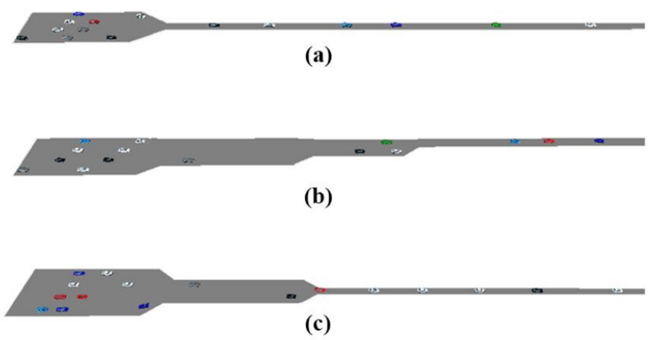


Figure 1: Merging Pattern

- An **one-point pattern** is where all lanes merge into one at the same point.

- A **rightmost (or Leftmost) pattern** is to always merge out the rightmost (or leftmost) lane until the desired number of lanes is reached.
- A **balanced pattern** is where the rightmost and leftmost lanes merge at the same point repeadly until the desired number of lanes is reached.

Accident prevention Accident prevention is measured by accident factor A. A certain type of toll area has its own accident factor. In consideration of features of traffic flows in toll plaza and reliabilities, we choose shape, queue length, and physiologic condition as three parameters to measure A. As is mentioned in the assumption, we neglect drivers' physiologic condition. In order to make A easy to be estimated and satisfy practical needs, we describe it as a linear equation[5].

$$\begin{aligned} A =& f(\theta, Q) \\ =& \beta_0 + \beta_1 Q + \beta_2 \tan\theta \end{aligned} \tag{2}$$

where Q is **Queue Length** and θ is **Shape**.

To figure out β_0, β_1, β_2, we assume sample capacity as N, $Y_i(i = 1, 2, 3...N)$ is Y value of the i^{th} sample. $N_{1i}, N_{2i}, N_{3i}...$ is value of variable in the i^{th} sample. Suppose that

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_N \end{bmatrix}, X = \begin{bmatrix} 1 & x_{11} & x_{21} & \dots & x_{N1} \\ 1 & x_{12} & x_{22} & \dots & x_{N2} \\ 1 & x_{13} & x_{23} & \dots & x_{N3} \\ 1 & \vdots & \vdots & & \vdots \\ 1 & x_{1N} & x_{2N} & \dots & x_{NN} \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_N \end{bmatrix} \tag{3}$$

then

$$\begin{aligned} Y =& X\beta \\ X^T Y =& X^T X \beta \end{aligned} \tag{4}$$

the maximum likelihood estimation of β is

$$\hat{\beta} = (X^T X)^{-1} X^T Y \tag{5}$$

then we obtain regression equation

$$Y = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \tag{6}$$

In this problem, we use the sample from $New\ Jersey\ Turnpike\ Authority$[6], and we obtain the matrix

$$\beta = \begin{bmatrix} 5.9475 \\ 3.0988 \\ -4.4012 \end{bmatrix} \tag{7}$$

then

$$A = 5.9475 + 3.0988Q + (-4.4012)\tan\theta \tag{8}$$

Throughput Throughput T represents number of vehicles per hour passing the point where the end of the plaza joins the L outgoing traffic lanes. The greater throughput, the easier it causes traffic congestion.

Cost Cost C means the money spent on land and road construction in merge area. Larger toll area means more expensive cost. We assume the cost of highway land construction is $\$x$ per square meter, and the cost to construct a tollbooth is $\$y$. Below is a schema of cost (Figure 2), as is shown in the schema, the width and the length of one tollbooth lane is a and b meters respectively.

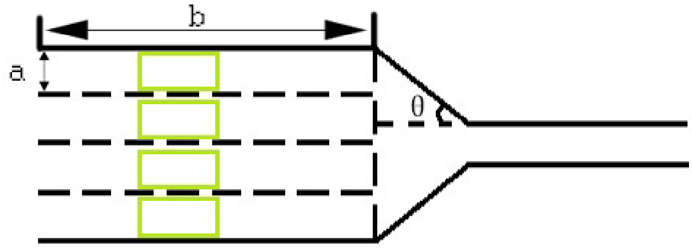


Figure 2: Schema of Cost when $S = \frac{1}{4}$

According to the assumptions we discussed in preceding section, we assume merging zone as ladder-shaped region with smooth lane.

$$\begin{aligned} C =& g(\theta, S) \\ =& \frac{[(S+1)a^2 \tan\theta + ab]x + y}{S} \end{aligned} \tag{9}$$

where S = $\frac{L}{B}$

Specially, in New Jersey, construct a new 4-lane highway - 4 million to 6 million per mile in rural and suburban areas, 8 million to 10 million per mile in urban areas, according to National Highway Construction Cost Index. [6]

Delay time Delay time is the difference between the real time of passing through and assumed time when the geometry is not affected by intersection, traffic conditions, control measures, such as factors affecting cases according to the original speed uniform through the time needed for crossing. It's noted as D.

This article uses the average intersection delay calculation method.The average calculation expression for the delay

$$u = \frac{0.5T(1 - \frac{t_g}{T})}{1 - [min(1, x) \cdot \frac{t_g}{T}]} \tag{10}$$

Where T stands for signal cycle length, $t_g$for effective green time, x saturation for road.

According to the statistical data and literature, we adopt the following index level:

Queue length The queue length at a certain period of time is the average number of retaining vehicles in studying area. It's noted as Q. It relates to retaining cars' occupation. Queue length is proportional to throughput T and average retaining time.[7]

Standard Level	1	2	3	4	5
u	[0,11)	[11,20)	[20,40)	[40,60)	> 60

Table 3: Intersection delay hierarchies suggestion value. Unit: s

4 Models

4.1 Markov Process

A Markov process is a specific type of a mathematical object known as a stochastic or random process, which is usually defined as a collection of random variables. [8]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t,\ t \in T)$, for some (totally ordered) index set T; and let $(S, \mathcal{S})$ be a measure space. An S-valued stochastic process $X = (X_t,\ t \in T)$adapted to the filtration is said to possess the Markov property with respect to the $\{\mathcal{F}_t\}$if, for each $A \in \mathcal{S}$ and each $s, t \in T$ with s < t,

$\mathbb{P}(X_t \in A|\mathcal{F}_s) = \mathbb{P}(X_t \in A|X_s)$.

In the case where S is a discrete set with the discrete sigma algebra and $T = \mathbb{N}$, this can be reformulated as follows:

$\mathbb{P}(X_n = x_n|X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0) = \mathbb{P}(X_n = x_n|X_{n-1} = x_{n-1})$.

N-steps transition probabilities

The n-steps transition probabilities satisfy the Chapman-Kolmogorov equation, that for any k such that 0 < k < n,[8]

$$p_{ij}^{(n)} = \sum_{r \in S} p_{ir}^{(k)} p_{rj}^{(n-k)} \tag{11}$$

where S is the state space of Markov process.

Markov process in this problem

Although in this problem, traffic flow is a continuous process, computer is only able to deal with discrete data. We approximate this continuous process through a great number of discrete data. The whole process is a Markov process. It has Markov property: for any positive integer r, n and $0 \le t_1 < t_2 < \cdots < t_r < m; t_i, m, n + m \in T_1$,

$$P(X_{m+n} = a_j|X_{t_1} = a_{i_1}, X_{t_2} = a_{i_2}, \cdots, X_{t_r} = a_{i_r}, X_m = a_i) = P(X_{m+n} = a_j|X_m = a_i) \tag{12}$$

where $a \in I$, and we noted the right side of the above formula as

$$P_{ij}(m, m + n) = P(X_{m+n} = a_j|X_m = a_i) \tag{13}$$

We call the Markov process state at the moment m is a_i, and the state at moment m+n is transferred to a_j. Matrix P (m, m+n) = $(P_{ij}(m, m + n))$ is the transition probability matrix of the Markov process.

4.2 Markov Process Model

The probability set of cars movement from one lane section to another is composed of Markov process. Cars movement at merging region is random walk. Random walk means a particle in a straight line (or plane or space) move in a certain region. In a traffic network, cars movement at the intersection between toll region and main lane is random walk.

Considering a common situation of lane network,suppose there are r starting point where cars come out or final point where cars vanish and s lane section. The Markov process transition probability represent of vehicles in the road network to represent the probability between elements, transition-probability matrix is

$$P_{(2r+s)\times(2r+s)} = \begin{bmatrix} I_{r\times r} & O_{r\times(r+s)} \\ R_{(r+s)\times r} & Q_{(r+s)\times(r+s)} \end{bmatrix} \tag{14}$$

where I is a unit matrix (r, r); O is a empty matrix (r, r+s) empty matrix; r said the vehicle from the point to the road or attraction transfer probability matrix (r+s, r); Q said the transfer probability matrix of vehicles in the event and between sections (r+s, r+s).

Since there is no direct transfer between the point of origin and the point of attraction, there is no vehicle to the point of occurrence, so $R_{(r+s)\times r}$and

$Q_{(r+s)\times(r+s)}$has the following characteristics:

$$R_{(r+s)\times r} = \begin{bmatrix} R_{1,r\times r} \\ R_{2,s\times r} \end{bmatrix} Q_{(r+s)\times(r+s)} = \begin{bmatrix} I_{r\times r} & O_{1,r\times s} \\ R_{s\times r} & Q_{2,s\times s} \end{bmatrix} \tag{15}$$

where $R_{1,r\times r}$ is an empty matrix, $R_{2,s\times r}$ is the transition probability from lane section to starting point. $O_{1,r\times s}$ is the transition probability from starting point to lane section; $O_{2,s\times;s}$ is the transition probability between lane sections; $O_{n,ij}$ represent cars starting from i to j by n step turn, where i, j represent lane section, starting point or final point..

For a lane network, toll number q = m, lane section number s = n, r^* is final point, r is starting point. The whole transition probability matrix is

$$R = \begin{bmatrix} & r_1^* & r_2^* & \cdots & r_m^* \\ r_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_m & 0 & 0 & \cdots & 0 \\ s_1 & s_1 r_1^* & s_1 r_2^* & \cdots & s_1 r_m^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_n & s_n r_1^* & s_n r_2^* & \cdots & s_n r_m^* \end{bmatrix} \tag{16}$$

$$Q = \begin{array}{cc} & \begin{array}{cccccccc} r_1 & r_2 & \cdots & r_m & s_1 & s_2 & \cdots & s_n \end{array} \\ \begin{array}{c} r_1 \\ \vdots \\ r_m \\ s_1 \\ \vdots \\ s_n \end{array} & \begin{bmatrix} 0 & 0 & \cdots & 0 & r_1s_1 & r_1s_2 & \cdots & r_1s_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & r_ms_1 & r_ms_2 & \cdots & r_ms_n \\ 0 & 0 & \cdots & 0 & s_1s_1 & s_1s_2 & \cdots & s_1s_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & s_ns_1 & s_ns_2 & \cdots & s_ns_n \end{bmatrix} \end{array} \tag{17}$$

where R is a (m+n, m) matrix; Q is a (m+n, m+n) matrix. Each element of the matrix represents the traffic flow transfer probability between the cell and the lane section.

4.2.1 OD Estimated Matrix Method

OD model is a model for the prediction of traffic flow. When we deal with the complicated network, not when the initial state vector is defined and evaluated using the OD model more accurately given initial state vector. Considering all the metastasis of OD, we get probability matrix produced from starting point to each road section by Chapman - the Kolmogorov equation [8]

$$(Q_{ij})_0 + (Q_{ij})_1 + (Q_{ij})_2 = \left[I - Q_{ij}\right]^{-1} \tag{18}$$

$$\begin{bmatrix} I & Q_{ij} \end{bmatrix}^{-1} = \begin{bmatrix} I_{r\times r} & -Q_{1,r\times s}[I_{s\times s} - Q_{1,s\times s}]^{-1} \\ O & [I_{s\times s} - Q_{1,s\times s}]^{-1} \end{bmatrix} \tag{19}$$

$[I - Q_{ij}]^{-1}R$ is a $(r+s, r)$ matrix, which represent probability of arriving at a final point from a starting point.

$$(Q_2)_0 + (Q_2)_1 + (Q_2)_2 + \cdots = [I - Q_2]^{-1} \tag{20}$$

$$U_{1\times r} = V_{1\times(r+s)}[I - Q_{ij}]^{-1}R_{(r+s)\times r} \tag{21}$$

$$x_{1\times s} = V_{1\times(r+s)}Q_{1\times(r+s)}[I - Q_{2,s\times s}]^{-1} \tag{22}$$

$$M_{OD} = V_{r\times(r+s)}[I - Q_{ij}]^{-1}R_{(r+s)\times r} \tag{23}$$

Formula 20 represents transition probability between each road section; $V_{r\times(r+s)}$ represent starting point traffic generation. $M_{OD}$is a OD matrix.

4.2.2 Algorithm

Algorithm 1 describes the general process of our Markov process model. For each experiment (different cars, different lanes, different tollbooths), we adjust parameters to

Markov matrix. Line 2 to 10 calculate the corresponding matrix at each step and finally get a queue length.

Algorithm 1: General Algorithm

```
   Data: Cars are represented as a Markov matrix;
   Result: matrix
1  Initialization;
2  while iteration is not zero do
3  |   starting point and final point are effective;
4  |   markov matrix multiples transition matrix;
5  |   if traffic jam then
6  |   |   starting point false ;          // means no cars comes
7  |   else
8  |   |   starting point true;
9  |   end
10 end
11 return matrix;
```

4.3 Model Performance

The number of zero-valued elements divided by the total number of elements is the density of the matrix (which is equal to 1 minus the density of the matrix)[9][10]. matrix density represents **Occupation**. The number of nonzero elements in certain rows means **Queue length**.

Iteration interval time

$$t_I = t_0 + \alpha\rho + \beta e^{k(\rho-0.5)} \tag{24}$$

where α, β and k are parameters. When $\rho < 0.5$, iteration time observes linear relationship, while $\rho > 0.5$ it relatively observes exponential relationship.

Overall time is T_{all}, iteration number is N_i

$$T_{all} = \sum_{n=1}^{N_i} t_I \tag{25}$$

Average time

$$T_{aveg} = \frac{T_{all}}{N_i} \tag{26}$$

In following paper, pattern 1, 2, and 3 refers to **one-point pattern**, **balanced pattern**, and **rightmost pattern** respectively. All the performances are detected in a certain size of matrix except from size change.

4.3.1 Size

We compare 3-Lane and 4-Lane module, the pattern and the type of car remain fixed. Results show that 4-Lane pattern 2 has a higher capacity of traffic. The cars move faster

	avg. time	avg. Quene	avg. delay
3LANE	15.2	14	2.5
4LANE p1	18.7	18	7.2
4LANE p2	12.6	2	1

Table 4: Size

and the queue length decreases in response. However, the 4-Lane pattern 1 is weak when compared with 3-Lane for worse cluster condition in fan-in area.

4.3.2 Shape

Shape θ may influence indexes including **accident factor**(A), **cost** and average pass time. Since all the factor have been defined in the third part, we can conclude the **Cost Function**

$$\arg\min_{\theta} E = T_{aveg} + \lambda C - \mu A \tag{27}$$

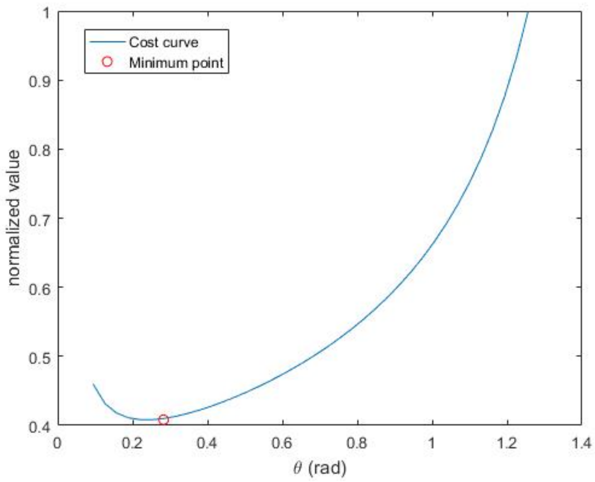


Figure 3: Vehicle Component

To simplify the Cost Function, we introduce a simple continuous model, omit constant and only concern about important variable. So we assume that the traffic is light and queue length equal to zero. According our collected data on NANJING TRANSPORT ADMINISTRATION (http://www.njjt.gov.cn/),we set λ and μ are 0.1 and -0.01 respectively. Because our matrix model cannot analyse this factor, we use model as figure2 and count time (T_{aveg}) directly from this pattern. $S = 1/4$, $a = 6$m, $b = 10$m and $v = 16.67$ m/s (60 km/h).

The equation can be rewritten as follows:

$$\begin{aligned}\arg\min_{\theta} E &= T_{aveg} + \lambda C - \mu A \\ &= \frac{a-S}{vS\sin\theta} + 0.1\frac{(S+1)a^2\tan\theta + ab}{S} - 0.01 \times (5.9475 - 4.4012\tan\theta)\end{aligned} \tag{28}$$

Cost Function has been normalized. From the figure we can see that E reaches its minimum point at $\theta = 19\pi/200$. So we take $\theta = 17^\circ$ as the best shape.

4.3.3 Pattern

Then we assess pattern effect. As the block shows above, apparently the one-point pattern gets the worst result, while the rightmost pattern has the least pass time and the maximum capacity of the traffic and the balanced one get the second pass time and high capacity. From the demo figure, pattern 3 has the maximum

	avg time	avg. quene	avg. delay
Pattern 1	18.7	18	7.2
Pattern 2	12.6	2	1
Pattern 3	12.4	0	0.7

Table 5: Pattern

4.4 A Better Solution – Dislocated Merging Zone

In preceding sections we discussed a series of models, and the results showed us better choices of shape, size, and merging pattern. Based on the results above, we raise a better solution named **'Dislocation Merging Zone'**. Seeing that the speciality of our solution, we change the size of our matrix which may lead to a slight difference between this outcome and experiments had been done before.

	avg time	avg occupation (%)
Better Solution	38.2	25.1
Common	39.3	24.7

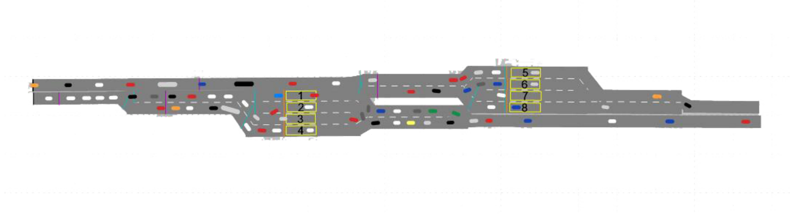
Table 6: Structure

In this solution we combine all the better choices in preceding tests of sub-models. We test our model when size $S = 3$ and $S = 4$, and we can draw a conclusion from the result that when $S = 4$, our model perform better. In our shape model, $\theta = \frac{\pi}{10}$ present a excellent balance between accident prevention and cost. When it comes to merging pattern, although rightmost pattern performs a little better than balanced pattern, it is less friendly in contraction efficiency. Parameters chosen are shown in the following table:

In real world, toll booths in many toll plazas are designed to align, we consider a solution dislocating several merging zone unit (see figure 4), and it is obvious that our solution make a better use of land. It will be tested in following sections.

size($S = \frac{L}{B}$)	shape(θ)	Merging Pattern
$\frac{1}{4}$	$\frac{\pi}{10}$	Balanced Pattern

Table 7: Parameters Chosen



(a) 2-D

(b) 3-D

Figure 4: Better Solution

5 Model Extension and Sensitivity Analysis

5.1 Traffic Load

As shown in figure 5, though our solution performs a little bit worse when traffic flow is between 2000 and 4000 per hour, it performs better than common one in light traffic ($<$ 2000 per hour). In heavy traffic(> 4000 per hour), our model performs much better than ordinary one.

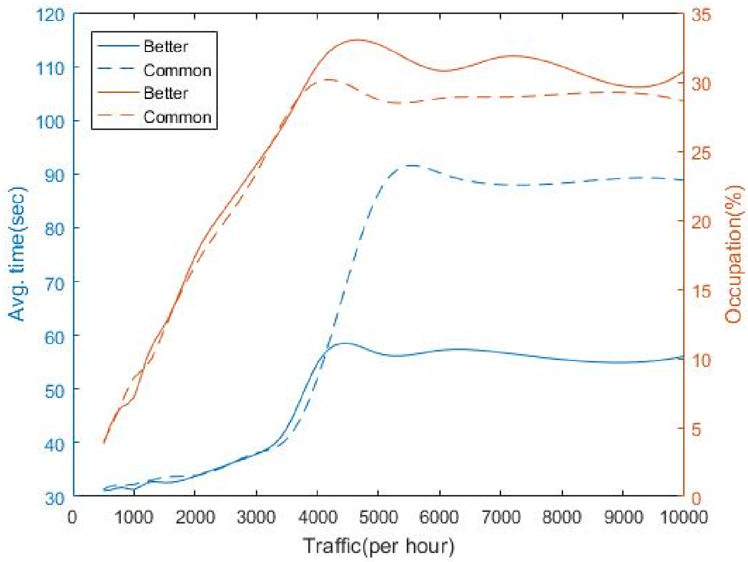
Sensitivity Test We change throughput T (traffic) to test its **optimal index** based on average time and occupation, as shown in figure 5 (b).Optimal index represents our solution's robustness, namely how insensitive it is to parameters' change.)[11]

Our solution's robustness is higher when T is larger than 5000 and optimal index is greater than 35%. Our solution's optimal index reaches 15% in light traffic.

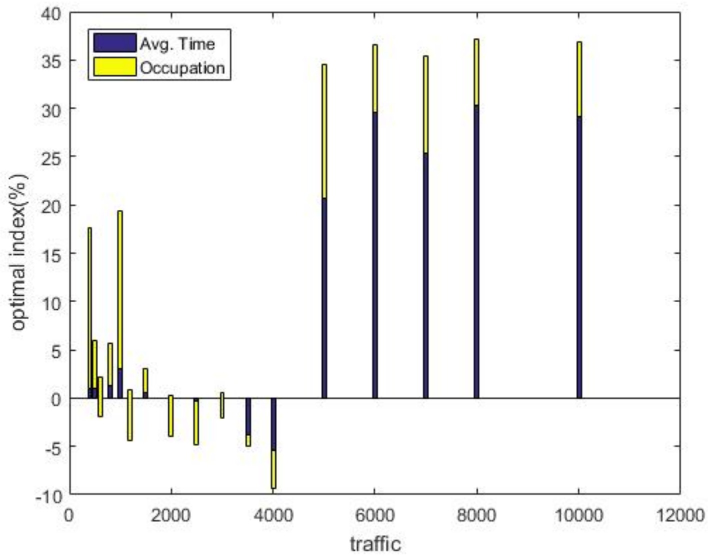
It illustrates that our model works quite well in relatively light traffic and heavy traffic.

5.2 Vehicle Type

Autonomous cars are free from emotional factor, typically cost more but well arranged and fast in speed. Meanwhile, a conventional car is cheap than a self-driving car, but is easily affected by driver condition.Since autonomous cars are faster, but keep at a longer distance from other ones, we simplify the model of both and only take specific factor **Avg.time** and **density** ρ and **Accident Index** into consideration.



(a) Better Model



(b) Optimal Index

Figure 5: Performance in light and heavy traffic

The relationship between ratio of autonomous car α and ρ:

$$\rho = \frac{\rho_0}{1+\alpha} \tag{29}$$

When $\alpha = 0$, $\rho = \rho_0$; $\alpha = 1$, $\rho = \rho_0/2$.

We use optimize function J to calculate the solution of our model. Accident Index(A) is different for autonomous or traditional cars separately, but indeed it does not change much so we take it as a constant.

$$\arg\max_{\alpha} J = \rho(\alpha) + \lambda T_{aveg}(\rho(\alpha))^{\gamma} + \mu A \tag{30}$$

Where λ, μ, γ are parameters depending on lane condition and the importance of three factors. γ is set to -0.5 empirically.

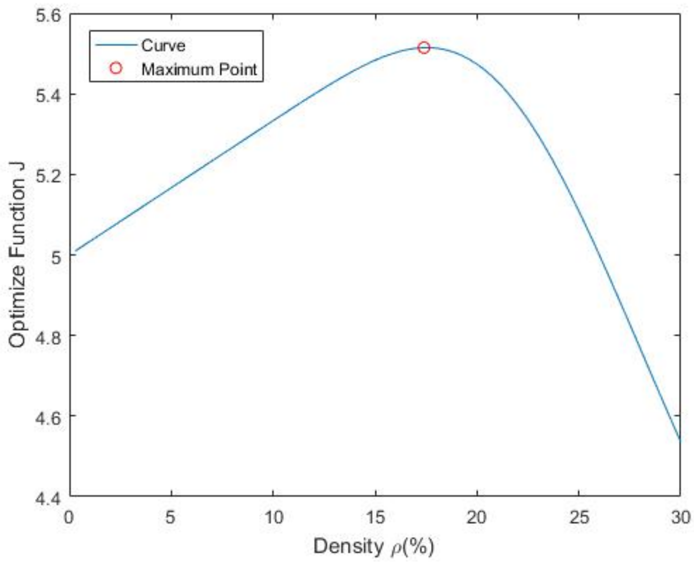


Figure 6: Vehicle Component

The maximum point $\rho = 19\%$, which means

$$\alpha = \frac{\rho_0}{\rho} - 1$$

For instance, if $\rho_0 = 25\%$, and then we get $\alpha = 32\%$.

When α goes up, by taking advantage of formula 29, ρ goes down. This means density reduced. We need to reduce merging zone size and then enlarge merging zone size. We make this change when we achieve the peak of density as shown in figure 6.

5.3 Tollbooths

$Y_1$is cost factor while Y_2 is traffic factor, which are normalized values by utilizing x, y. x is cost attribution and y is traffic attribution, which are got from above experiments.

α is the proportion of different types of tollbooths. α_1, α_2 ,α_3 represents Conventional tollbooth proportion, Exact-change tollbooth proportion, Electronic tollbooth proportion respectively.

$$\begin{aligned} Y_1 =& \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \\ Y_2 =& \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \end{aligned} \tag{31}$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$

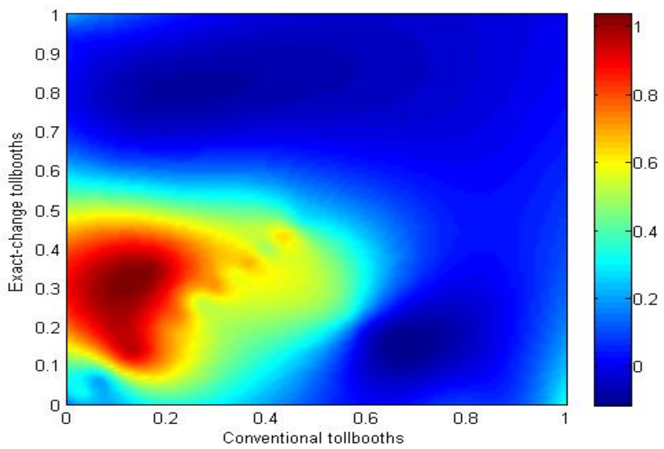
Y is tollbooth index, which is also a normalized value.

$$
\begin{aligned}
Y =& \lambda_1 Y_1 + \lambda_2 Y_2 \\
=& \alpha_1(\lambda_1 x_1 + \lambda_2 y_1) + \alpha_2(\lambda_1 x_2 + \lambda_2 y_2) + \alpha_3(\lambda_1 x_3 + \lambda_2 y_3)
\end{aligned} \tag{32}
$$

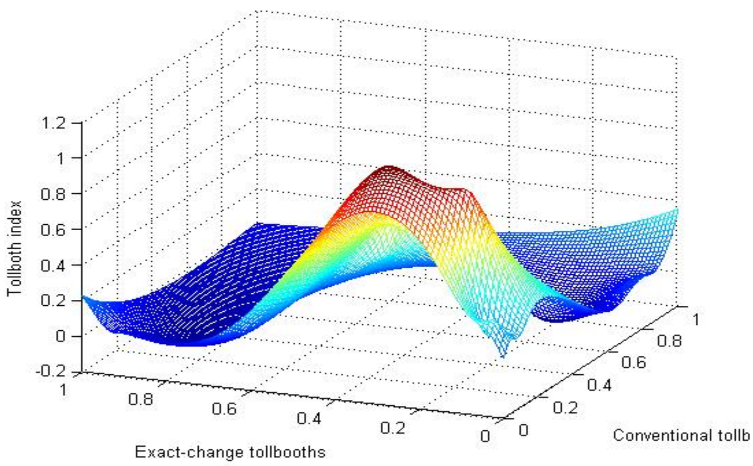
where $\lambda_1 + \lambda_2 = 1$

We change different proportion of tollbooths and get figure 7. It is impossibile that conventional tollbooths and Exact-change tollbooths $\alpha_1 + \alpha_2 > 1$. We don't have any experiment point in this regioin $\alpha_1 + \alpha_2 > 1$, so the tollbooth index is 0 here.

From figure 7 (a), the best tollbooth proportion is in the red region.



(a) Density



(b) 3D

Figure 7: Tollbooth

Optimization

This is a convex problem, and we can find optimal value. Through gradient ascent optimization[12], We take steps proportional to the positive of the gradient, and then approach a local maximum of that function, as shown in formula 33. This local maximum is the overall maximum of our function. We find the optimal solution, as shown in table 8.

$$x_1 = x_0 + \frac{\partial y}{\partial x} \tag{33}$$

Type	Conventional tollbooth	Exact-change tollbooth	Electronic tollbooths
Proportion	0.18	0.34	0.48

Table 8: The Best Proportion

6 Strengths and Weakness

6.1 Strengths

- We take various factors into account, like delay time and queue length, which make our model more close and adaptable to a real turnpike system.
- We find a innovative dislocated structure and it is tested out to be outstanding.
- Our model comes from a series of sub-models and ease some main and serious problems in the real world like heavy cost and traffic jam.
- Our model withstands heavy traffic condition.
- Our model shows low sensitivity in variation of parameters.
- We made extensions to our model that takes various cars, lanes into account.

6.2 Weakness

- If our model reacting as consultants for a particular highway, more information like topography data and weather data should be collected and measured.
- We assume that drivers' physiologic condition is negligible, but actually, drivers' condition has great impact on operation of turnpike.
- In the real world, a car in the center lane has an easier time merging into the center lanes than a car in a peripheral lane, but this behavior is not reflected in our model.
- All cars behave the same and all tollbooth lanes are homogeneous. We neglected those difference happens in production progress.

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Letter to New Jersey Turnpike Authority

Dear New Jersey Turnpike Authority,

We are a team in 2017 MCM contest, and we are writing to give an advice on merging zone construction.

Many cars pass by New Jersey highway to reach other destinations, therefore, tolls are basic requirements to maintain its repair. New Jersey is a dense turnpike owner, but a great number of toll plazas and uneven lanes account for a higher rate of traffic accidents and traffic congestion. A better solution for merging zone construction is desperately needed.

We care a lot about the problem from different perspective. We construct models to depict merging zone and cars flow and finally come up with a unused solution, which named $Dislocated\ Merging\ Zone$. Tests show that it performs much better than some widely-used solutions.

First, let us explain our solution. Just as the name implies, $Dislocated\ Merging\ Zone$ means a combination of several dislocated merging zone units. A vivid explanation is shown in the figure below:

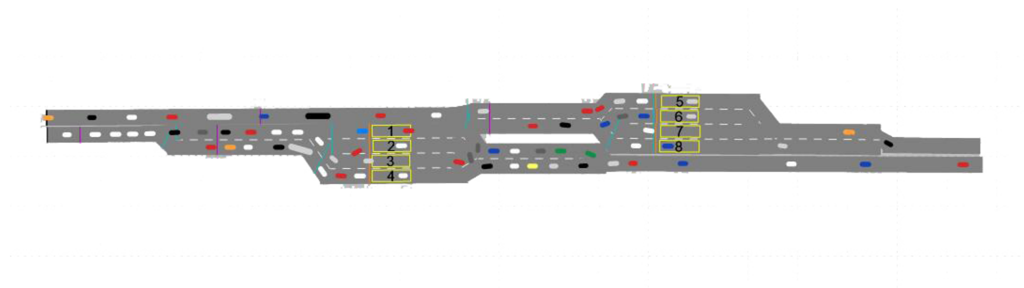

Figure 8: Dislocated Merging Zone

We only combine two units of merging zone in our test, with balance merging pattern and the ratio of number of highway lane and toll lane is $\frac{1}{4}$.

Our test shows that $Dislocated\ Merging\ Zone$ lower cars' average pass time and it performs extremely well in heavy traffic compared with common used merging zone, which means it is able to ease traffic jam in toll plazas. And obviously, it makes better use of the land and cost less. So we think our model can be a good choice in merging zone construction.

We also make some extension in our model. As more and more self-driving car adding to traffic mix, the utilization of lanes lowers because of features of self-driving car. So we highly suggest that peripheral lanes with electronic tollbooths specially use by self-driving car, in this way, self-driving cars and ordinary cars can be divided to avoid mutual influence. What's more, we found that our model works more efficient when the proportion of manual tollbooths, automated tollbooths and electronic tollbooths is 18%, 34% and 48% respectively. So more electronic tollbooths should be put into use and more cars information should be input into electronic tollbooths system. Finally, our model doesn't require much about the width of land, so it is adaptive in some rural area where there is no much vast land.

We strongly recommend our model to you, and more real-world condition should be take into account before it is put into use. And we sincerely hope that our model will do some help in solving the problems.

Best regards